ASSIGNMENT SET – I

Mathematics: Semester-I

M.Sc (CBCS)

Department of Mathematics

Mugberia Gangadhar Mahavidyalaya

PAPER - MTM-103

Paper: Ordinary Differential Equations and Special Functions

1. Define fundamental set of solutions for system of ordinary differential equation.
2. What is meant by singularity of a linear ordinary differential equation?
3. Find the regular singular point of the ODE : $x(x-1)y''+(\sin x)y'+2x(x-1)y=0.$
4. Discuss Frobenious method of finding the series solution about the regular singularpoint at the origin for an ODE of 2nd order when the roots of the indicial equation are equal.
5. Examine that whether infinity is a regular singular point for Bessel's differential equation or not.
6. Show that the eigen functions of the regular Sturm-Liouville system $\frac{d}{dx}[p(x)\frac{dy}{dx}]+[q(x)+\lambda r(x)]y=0$ having different eigen values are orthogonal with respect to weight function r(x).
7. All the eigen values of regular SL problem with $r(x)>0,$ are real.
8. Prove that F(-n, b, b, -z)=$(1+z)^2$ where F(a, b, c, z) denotes the hypergeometric function.
9. Deduce the integral formula for hypergeometric function.
10. Find the integral formula for hypergeometric function with the necessary condition.
11. Under suitable transformation to be considered by you , prove that Legendre differentialequation can be reduced to hypergeometric equation.
12. Prove that for the hypergeometric function $F(\alpha,\beta,\gamma,z)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha}\,dt.$

13. Find the series solution near z=0 of $(z+z^2+z^3)\ddot{W}(z)+3z^2\dot{W}(z)-2W(z)=0$
14. Let $P_n(z)$ be the Legendre's polynomial of degree n and $P_{m+1}(0)=\frac{-m}{m+1}P_{m-1}(0), m=1,2,3...$. If $P_n(0)=\frac{-5}{16}$, then find the value of $\int_{-1}^{1}P_n^2(z)dz.$
15. Find the general solution of the ODE $2zw''(z)+(1+z)w'(z)-kw=0.$ (where k is a real constant) in series form. For which values of k is there a polynomial solution?
16. Define INDICAL equation in connection with Frobenious method.
17. What do you mean by INDICIAL equation connecting ODE?
18. Find all the singularities of the following differential equation and then classify them: $(z-z^2)^2\omega''+(1-5z)\omega'-4\omega=0.$
19. Define a self-adjoint differential equation with an example.
20. Find all the singularities of the following differential equation and then classify: $z^2(z^2-1)^2\omega''-z(1-z)\omega'+2\omega=0.$
21. Define fundamental set of solutions for system of ordinary differential equation.
22. Let W(f, g) be the wronskian of two linearly independent solutions f and g of the equation $\ddot{W}+P(z)\dot{W}+Q(z)W=0$. Then find the value of product of W(f, g)P(z).
23. Find the general solution of the homogeneous equation $\frac{d\vec{x}}{dt}=\begin{pmatrix}1 & -2 & 0\\ 2 & 3 & 0\\ 0 & 0 & 2\end{pmatrix}\vec{x}$ where $\vec{x}=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix}$
24. Define fundamental set of solutions and fundamental matrix for system of differential equations.

25. Show that o is the regular singular and 1 is the irregular singular points of the differential equation $(z-1)\ddot{W}(z)+(\cot\pi z)\dot{W}(z)+\operatorname{cosec}^2\pi z\,W(z)=0$

26. Show that $\frac{dy}{dx}=3y^{\frac{2}{3}}, y(0)=0$ has more than one solution and indicate the possible reason.

27. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solution of $x^2\ddot{y}-2x\dot{y}-4y=0$,forall x in [0,10] *consider the Wronskian* $W(x)=y_1(x)\,y_2'(x)-y_1'(x)\,y_2(x)$. *If* $W(1)=1$ then find the value

of W(3)$-W(2)$?

28. Find the series solution near z=0 **of** $2z^2W''(z)+zW'(z)-(z+1)W(z)=0$

Using the fact that $y=x^2$ is a solution of the ordinary differential equation $x^2\dfrac{d^2y}{dx^2}-3x\dfrac{dy}{dx}+4y=0, x>0$, find the another independent solution.

35. Find the steady state point of $\dfrac{dy}{dx}=e^{x-1}-1$ and $\dfrac{dy}{dx}=ye^x$

36. Discuss about the stability of the following system of differential equations;

$\dfrac{dy}{dx}=-x+y$ and $\dfrac{dy}{dx}=4x-y$

37. Discuss about the stability of the following system of

differential equations; $\dfrac{dy}{dx}=x+2y$ and $\dfrac{dy}{dx}=x^2+y$

38. Show that f(x, y)=xy^2 on ℝ : { |x|≤1, |y|≤1} satisfying a Lipschitz condition. But this function does not satisfy a Lipschitz condition on the strip S: { |x|≤1, |y|≤∞}.

39. Solve the following system of differential equations;

$$\frac{dy}{dx}+4x+3y=t, \qquad \frac{dy}{dx}+2x+5y=e^t.$$

40. Find the fundamental matrix and complementary solution of the homogenous linear system of differential equations :

$$\frac{dx}{dt}=3x+y, \qquad \frac{dy}{dt}=x+3y.$$

41. Solve : $(1+3x)^2\frac{d^2y}{dx^2}-6(1+3x)\frac{dy}{dx}+6y=8(1+3x)^2$ $,-\frac{1}{3}<x<\infty$

42. Find the power series solution of the equation $4x^2y''(x)+2xy'(x)-(x+4)y=0$ in power of x.

42. Show that the solution of the ODE $\frac{dx}{dt}=2x+y$ & $\frac{dy}{dt}=3x$ satisfy the relation$3x+y=Ke^{3t}$where K is a real constant.

43. Find the equilibrium point for the system of equations $\dot{x}=2x+7y$ and $\dot{y}=x^2+3y$.

44. Find the ordinary &singular point of the ODE $3x^2y'' + 7x(x+2)y' - 4y = 0$.

45. State the picard theorem for existence and uniqueness for $\frac{dy}{dx} = f(x,y)$ with $y(x)=y_0$. Also show for $\frac{dy}{dx} = \frac{1}{y}, y(0)=0$, has more than one solution indicate the possible reason.

46. $y(x) = x^2 \sin x$ is a solution of an n-th order linear ODE $y^n(x)+a_1y^{(n-1)}(x)+\ldots\ldots\ldots+a_{(n-1)}y(x) + a_ny(x) = 0$ with real constant coefficients, then prove that the least possible value of n is 4.

47. What do you mean by Wronskian in ODE and state its utility?

48. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solution of

$x^2\ddot{y} - 2x\dot{y} - 4y = 0$, for all x in [0,10] $consider the\, Wronskian$ W(x) = $y_1(x)y_2'(x) - y_1'(x)\, y_2(x)$. $If W(1)$ =1 then find the value of $W(3) - W(2)$?

49. Let $w_1(z)$ and $w_2(z)$ be two solutions of $(1-z^2)w''''(z) - 2zw''(z) + (\sec z)w = 0$ with Wroinskian w(z). If $w_1(0) = 1$, $w''(0) = 0$, and w $(\frac{1}{2}) = \frac{1}{3}$, then find the value of $w_2'(z)$ at z=0.

50. Show that $n\,P_n(z) = zP'_n(z) - P'_{n-1}(z)$, where $P_n(z)$denotes the Legendre polynomial of degree n.

51. Deduce Rodrigue's formula for Legendre's polynomial.

52. Show that $nP_n(z) = zP_n'(z) - P_{n-1}'(z)$, where $P_n(z)$denotes the Legendre polynomial of degree n.

53. Under suitable transformation to be considered by you , prove that Legendre differential equation can be reduced to hypergeometric equation.

54. Prove that $\int_{-1}^{1} P^2{}_n(z)dz = \frac{2}{2n+1}$. where $P_n(z)$ is the Legendre's Polynomial of degree n.

55. Examine that whether infinity is a regular singular point for Legendre's differential equation or not.

55. Let the Legendre equation $(1-z^2)\ddot{W}(z) - 2z\dot{W}(z) + n(n+1)W(z) = 0$ have n-th degree polynomial solution $W_n(z)$ such that $W_n(1) = 3$ If $\int_{-1}^{1}(W_n^2(z) + W_{n-1}^2(z))dz = \frac{16}{15}$ then find the value of n.

56. Prove that if $f(z)$ is continuous and has continuous derivatives in $[-1,1]$. Then $f(z)$ has unique Legendre series expansion is given by $f(z) = \sum_{n=0}^{\infty} c_n P_n(z)$ where $P_n's\ are\ Legendre\ polynomials\ c_n = \frac{2n+1}{2}\int_{-1}^{1} f(z)P_n(z)dz, n = 1,2,3 \ldots\ldots$

57. Show that $n\,P_n(z) = zP'_n(z) - P'_{n-1}(z)$, where $P_n(z)$denotes the Legendre polynomial of degree n.
58. Show that $1 + 3P_1(z) + 5P_2(z) + 7P_3(z) + \cdots + (2n+1)P_n(z) = \frac{d}{dz}[P_{n+1} + P_n(z)]$Where $P_n(z) denotes the Legendre's$ polynomial of degree n
59. Establish generating function for Legendre polynomial. Use it to prove that

    $(2n+1)zP_n(z) = (z+1)P_{n+1}(z) + nP_{n-1}(z)$

60. Let $P_n(z)$ be the Legendre polynomial of degree n such that $P_n(1) = 1$, n= 1,2,3,….
61. Let $P_n(z)$ be the Legendre's polynomial of degree n and

    . $P_{m+1}(0) = \frac{-m}{m+1}P_{m-1}(0), m = 1,2,3 \ldots\ldots\ldots$ If $P_n(0) = \frac{-5}{16}$, then find the value of

    $\int_{-1}^{1} P_n^2(z)dz$.

62. Establish the Bessel integral equation.
63. Prove that $\frac{d}{dz}[z^{-n}J_n(z)] = -z^{-n}J_{n+1}(z)$ Where $J_n(z)$ is the Bessel ' s function.
64. Examine that whether infinity is a regular singular point for Bessel's differential equation or not.
65. What are Bessel's functions of order n? State for what values of n the solutions are independent of Bessel's equation of order n.
66. Establish the generating function for the Bessel's function $J_n(z)$.
67. Show that $J_n(z)$ is an odd function of z if n is odd.
68. If α and β are the roots of the equation $J_n(z)=0$, then show that

    $$\int_0^1 zJ_n(\alpha z)J_n(\beta z)dz = \begin{cases} 0, & \alpha \neq \beta \\ \frac{1}{2}[J'_n(z)]^2, & \alpha = \beta \end{cases}$$

69. Show that when n is a positive integer, $J_n(z)$ is the coefficient of z^n in the expansion of $\exp\{\frac{z}{2}(t - \frac{1}{t})\}$ i.e. $\exp\{\frac{z}{2}(t - \frac{1}{t})\} = \sum_{n=-\infty}^{n=\infty} t^n J_n(z)$
70. Show that when n is a positive integer, $J_n(x)$ is the coefficient of z^n in the expansion of $\exp(\frac{x(z - \frac{1}{z})}{2})$.
71. Prove that for the Bessel's function $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$

72. Show that $J_0^2(z) + 2\sum_{n=1}^{\infty} J_n^2(z) = 1$ and prove that for real z, $|J_0(z)| < 1$ and $|\boldsymbol{J_n(z)}| < \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}$**,** for all $n \geq 1$ **.**

73. Prove that $\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$.

74. Show that $J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}} cosx.$

75. Show that when n is a positive integer, $J_n(z)$ is the coefficient of z^n in the expansion of $\exp\{\frac{z}{2}(t-\frac{1}{t})\}$ i.e. $\exp\{\frac{z}{2}(t-\frac{1}{t})\} = \sum_{n=-\infty}^{n=\infty} t^n J_n(z)$

76. Discuss Frobenious method of finding the series solution about the regular singularpoint at the origin for an ODE of 2nd order when the roots of the indicial equation are equal.

77. Show that x =0 is a ordinary point and x =2 is a regular singular point of the ODE $x(x-2)y'' + (sinx)y' + 2x(x-2)y = 0.$

78. Examine that whether infinity is a regular singular point for Bessel's differential equation or not.

79. Prove that F(-n, b, b, -z)=$(1+z)^2$ where F(a, b, c, z) denotes the hypergeometric function.

80. Deduce the integral formula for hypergeometric function.

81. Find the integral formula for hypergeometric function with the necessary condition.

82. Find the series solution near z=0 of $(z+z^2+z^3)\ddot{W}(z)+3z^2\dot{W}(z)-2W(z)=0$
Find the general solution of the ODE $2zw''(z) + (1+z)w'(z) - kw = 0.$ (where k is a real constant) in series form. For which values of k is there a polynomial solution?

83. Find all the singularities of the following differential equation and then classify them: $(z-z^2)^2\omega'' + (1-5z)\omega' - 4\omega = 0.$

84. Define a self-adjoint differential equation with an example.

85. Find all the singularities of the following differential equation and then classify:

$$z^2(z^2-1)^2\omega''-z(1-z)\omega'+2\omega = 0.$$

86. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solution of

$x^2\ddot{y} - 2x\dot{y} - 4y = 0$,forall x in [0,10] $considertheWronskian$ W(x) = y_1(x)$y_2{}'$(x)−$y_1{}'$(x) y_2(x). $IfW(1)$ =1 then find the value of W(3)−$W(2)$?

87. Let $w_1(z)$ and $w_2(z)$ be two solutions of $(1-z^2)w''''(z) - 2zw''(z) + (\sec z)w = 0$ with Wroinskian w(z). If $w_1(0) = 1$, $w''(0) = 0$, and w $(\frac{1}{2})$ =$\frac{1}{3}$, then find the value of $w_2'(z)$ at z=0.

88. Write the important properties of Green's function related to ordinary differential equation.
89. Show that the Green's function of a given problem is everywhere continuous.
90. Using Green's function method ,solve the following differential equation$\boldsymbol{y'''(x) = 1}$ Subject to boundary $conditionsy(0) = y(1) = y'(0) = y'(1)$.
91. Using Green's function method solve the equation $\frac{d^2y}{dx^2} + y = x^2, y(0) = y(\frac{\pi}{2}) = 0$
92. Define Green's function of the differential operator L of the non- homogeneous differential equation: Lf(x)=f(x).
93. Using Green's function method solve the equation $\frac{d^2X}{dt^2} = f(x)$ subject to the conditions y(0)=0, y(a)=0.
94. Define Green's function of the differential operator L of the non-homogeneous differential equation : Lf(x) = f(x) **.**
95. All the eigen values of regular SL problem with $r(x) > 0,$ are real.
96. Write the importance of Sturm-Liouville problems.
97. When a boundary problem is a Sturm-Liouville problem?
98. Find the values of a and b for which the boundary value problem $x^2y'' - 2xy' + 4y = 0$ subject to the boundary condition y(1) + a$y'(1) = 1$ and y(2)+b$y'(2) = 2$ has a unique solution.
99. Define the orthogonal functions.
100. Show that the eigen functions of the regular Sturm-Liouville system $\frac{d}{dx}[p(x)\frac{dy}{dx}] + [q(x) + \lambda r(x)]y = 0$ having different eigen values are orthogonal with respect to weight function r(x).
101. When a boundary problem is a Sturm-Liouville problem?
102. An n-th order ODE is equivalent to a system of n first order ODEs.Prove that

$$F(\alpha,\beta,\gamma,x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt$$

103. Find the values of a and b for which the boundary value problem

$x^2y'' - 2xy' + 4y = 0$ subject to the boundary condition y(1) + a$y'(1) = 1$ and y(2)+b$y'(2) = 2$ has a unique solution.

104. Consider the boundary value problem

$$\frac{d^2y}{dx^2} + \lambda y = 0, 0 \le x \le \pi$$

Subject to y(0)=0,y$(\pi) = 0$. $Find the values of \lambda$ for which the boundary value problem is solvable.

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